Travelling waves for
the nonlinear Schrödinger equation

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Introduction

1. The nonlinear Schrödinger equations

The nonlinear Schrödinger equations under consideration write as

\[ i \partial_t \psi + \Delta \psi + \psi f(|\psi|^2) = 0, \quad \text{(NLS)} \]

for a function \( \psi : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C} \) and a nonlinearity \( f : \mathbb{R} \to \mathbb{R} \) such that

\[ f(1) = 0 \quad \text{and} \quad f'(1) < 0. \]

A typical example is the Gross-Pitaevskii equation given by

\[ i \partial_t \psi + \Delta \psi + \psi (1 - |\psi|^2) = 0, \quad \text{(GP)} \]

for \( f(\tau) = 1 - \tau \).
The (NLS) equation is Hamiltonian. Its energy is given by

\[ E(\Psi) = \frac{1}{2} \int_{\mathbb{R}^N} \left( |\nabla \Psi|^2 + V(|\Psi|^2) \right), \]

where the potential \( V \) is equal to

\[ V(\tau) = \int_{\tau}^1 f(\sigma) \, d\sigma. \]

For (GP), this expression is exactly the Ginzburg-Landau energy. At least formally, solutions \( \Psi \) with finite energy satisfy a non vanishing condition at infinity

\[ |\Psi(x)| \to 1, \quad \text{as } |x| \to \infty. \]
The (NLS) equation is dispersive. Its linearization around the constant solution $\Psi = 1$ leads to the linear equation
\[ i \partial_t \varepsilon + \Delta \varepsilon + 2f'(1)\text{Re}(\varepsilon) = 0. \]
The dispersion relation is equal to
\[ \omega^2 = -2f'(1)|k|^2 + |k|^4. \]
The sound speed is
\[ c_s = \sqrt{-2f'(1)}. \]
The (NLS) equation also owns an hydrodynamical formulation. When the solution $\Psi$ can be expressed in terms of the Madelung transform

$$\Psi = \sqrt{\rho} e^{i\theta},$$

the hydrodynamical variables $(\rho, v = 2\nabla \theta)$ satisfy the hydrodynamical system

$$\begin{cases}
\partial_t \rho + \text{div}(\rho v) = 0, \\
\partial_t v + v \cdot \nabla v - 2\nabla f(\rho) = 2\nabla \left( \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right). 
\end{cases}$$

(HGP)
2. Travelling waves

**Travelling waves** are special solutions of the form

\[ \Psi(x, t) = U_c(x_1 - ct, \ldots, x_N), \]

for speeds \( c \in \mathbb{R} \). Their profile \( U_c \) is solution to the nonlinear elliptic equation

\[ -ic\partial_1 U_c + \Delta U_c + U_c f(|U_c|^2) = 0. \] (1)
In dimension $N = 1$, the (non constant) travelling waves for (GP) are uniquely given by

$$U_c(x) = \sqrt{\frac{2-c^2}{2}} \tanh \left( \frac{\sqrt{2-c^2}}{2} x \right) + i \frac{c}{\sqrt{2}},$$

for $|c| < \sqrt{2}$ (up to translation and phase shift).

Due to the integrability of the equation by means of the inverse scattering transform (Zakharov-Shabat [73]), the dark solitons $U_c$ are believed to play a major role in the long-time dynamics.
I. The Cauchy problem

**Theorem** (Zhidkov [01], Gérard [06], Killip-Oh-Pocovnicu-Visan [12]).

For $N \in \{1, 2, 3, 4\}$, let $\Psi_0$ be a function in the energy set

$$\mathcal{E}(\mathbb{R}^N) = \{\psi : \mathbb{R}^N \to \mathbb{C} \text{ s.t. } E(\psi) < +\infty\}.$$ 

There exists a unique global solution in $\mathcal{C}^0(\mathbb{R}, \mathcal{E}(\mathbb{R}^N))$ with initial datum $\Psi_0$ for (GP). Moreover, the Ginzburg-Landau energy is conserved along the flow.

See Gallo [04, 06] for (NLS).
II. Construction of (non constant) travelling waves

For (GP), Jones, Putterman and Roberts [82, 86] investigated the existence and qualitative properties of travelling waves in dimensions $N = 2$ and $N = 3$.

They claimed the non-existence of supersonic travelling waves and exhibited a smooth branch of subsonic travelling waves.
1. Non-existence of travelling waves

**Theorem** (Bethuel-Saut [99], G. [03, 04]).

For (GP), a travelling wave with speed $c = 0$ for $N \geq 2$, $c > \sqrt{2}$ for $N \geq 2$, and $c = \sqrt{2}$ for $N = 2$, is constant.

See Maris [08] for (NLS).

**Theorem** (Bethuel-G.-Saut [07], de Laire [08]).

Let $N \geq 3$. For (GP), there exists a number $\mathcal{E}_N > 0$ such that a travelling wave $U$ with energy

$$E(U) \leq \mathcal{E}_N,$$

is constant.
2. Minimizing travelling waves

For fixed $p$, minimizing travelling waves $U_p$ solve the variational problem

$$E_{\text{min}}(p) = \inf \left\{ E(u), u : \mathbb{R}^N \to \mathbb{C} \text{ s.t. } p(u) = p \right\}.$$ 

Here, the scalar momentum $p$ is the first component of the momentum, which is formally given by

$$p(u) = \frac{1}{2} \int_{\mathbb{R}^N} \langle i\partial_1 u, u \rangle_{\mathbb{C}}.$$ 

The speed $c_p$ is the Lagrange multiplier of the previous problem.
Theorem (Bethuel-G.-Saut [07]).

(i) For $N = 2$ and any $p > 0$, there exists a minimizing travelling wave $U_p$ for (GP).

(ii) For $N = 3$, there exists a critical value $p_\ast > 0$ such that there exists a minimizing travelling wave $U_p$ for (GP) if and only if

$$p \geq p_\ast.$$

(iii) The speed $c_p$ of the travelling wave $U_p$ satisfies

$$0 < \frac{dE_{\text{min}}}{dp}(p^+) \leq c_p \leq \frac{dE_{\text{min}}}{dp}(p^-) < \sqrt{2}.$$

**Theorem** (Chiron-Pacherie [19, 19, 21]).

(i) Let $N = 2$. There exists a number $p_0 > 0$ such that, for any $p \geq p_0$, the travelling waves $U_p$ for (GP) are unique (up to translation and phase shift). Moreover, they form a smooth branch of travelling waves.

(ii) In the limit $p \to 0$, their speed $c_p$ is of order

$$c_p \sim \frac{2\pi}{p},$$

and they own two vortices of degree $\pm 1$ at a distance $d_p$ of order

$$d_p \sim \frac{p}{\pi}.$$  

See also Bethuel-Saut [99], Bethuel-Orlandi-Smets [04] and Chiron [04].
Set $\Omega_L = \mathbb{R} \times T_L$ for any number $L > 0$.

**Theorem** (de Laire-G.-Smets [22]).

(i) Let $0 < p \leq \pi/2$. There exists a number $L_p > 0$ such that, for any $0 < L \leq L_p$, there exists a minimizing travelling wave $U_{pL}$ for (GP) on $\Omega_L$.

(ii) The travelling wave $U_{pL}$ only depends on the variable $x_1$. In particular, it is a dark soliton.

3. Subsonic travelling waves

**Theorem** (Maris [09]).

Let $N \geq 3$. There exists a non constant travelling wave $U_c$ of (NLS) for any speed $0 < c < \sqrt{2}$.

See also Bellazzini-Ruiz [19].
III. Orbital and asymptotic stability of travelling waves

1. In dimension $N = 1$

a. Orbital stability of dark solitons

Let us endow the energy set $\mathcal{E}(\mathbb{R})$ with the distance

$$d_c(\psi_1, \psi_2)^2 = \int_{\mathbb{R}} |\psi_2' - \psi_1'|^2 + (1 - |U_c|^2)|\psi_2 - \psi_1|^2 + \left| |\psi_1|^2 - |\psi_2|^2 \right|^2.$$
Theorem (Bethuel-G.-Saut [08], Bethuel-G.-Saut-Smets [08]).
Let $c \in (-\sqrt{2}, \sqrt{2})$. There exist two numbers $\delta_c > 0$ and $K_c > 0$ such that, if an initial datum $\Psi^0 \in \mathcal{E}(\mathbb{R})$ satisfies the condition

$$\delta := d_c(\Psi^0, U_c) < \delta_c,$$

then there exist two functions $a \in C^1(\mathbb{R}, \mathbb{R})$ and $\theta \in C^1(\mathbb{R}, \mathbb{R})$, with

$$\sup_{t \in \mathbb{R}} |a'(t) - c| < K_c \delta,$$

such that the corresponding solution $\Psi$ for (GP) satisfies

$$\sup_{t \in \mathbb{R}} d_c\left(e^{-i\theta(t)}\Psi(\cdot + a(t), t), U_c\right) < K_c \delta.$$

See also Gérard-Zhifei Zhang [08], and Chiron [13] for (NLS).
b. Asymptotic stability of dark solitons

**Theorem** (Bethuel-G.-Smets [13], G.-Smets [14]).

Let $c \in (-\sqrt{2}, \sqrt{2})$. There exists a number $\delta_c > 0$ such that, if the initial datum $\Psi^0 \in \mathcal{E}(\mathbb{R})$ satisfies the condition

$$d_c(\Psi^0, U_c) < \delta_c,$$

then there exist a number $c_\infty \in (-\sqrt{2}, \sqrt{2})$, and two functions $a \in C^1(\mathbb{R}, \mathbb{R})$ and $\theta \in C^1(\mathbb{R}, \mathbb{R})$, with

$$a'(t) \to c_\infty \quad \text{and} \quad \theta'(t) \to 0,$$

as $t \to +\infty$, such that the corresponding solution $\Psi$ for (GP) satisfies

$$e^{-i\theta(t)}\Psi(\cdot + a(t), t) \to U_{c_\infty} \quad \text{locally uniformly on } \mathbb{R}. $$

See also Cuccagna-Jenkins [16].
2. In higher dimensions

For $N = 2$ and $N = 3$, the energy set $\mathcal{E}(\mathbb{R}^N)$ can be endowed with the distance

$$d(f, g) = \|f - g\|_{L^2(B(0,1))} + \|\nabla f - \nabla g\|_{L^2(\mathbb{R}^N)} + \|f^2 - g^2\|_{L^2(\mathbb{R}^N)}.$$

**Theorem** (Chiron-Maris [11]).

Let $\mathcal{M}_p$ be the set of minimizing travelling waves $U_p$ with scalar momentum $p$ (with $p \geq p_*$ if $N = 3$).

Fix a travelling wave $U_p \in \mathcal{M}_p$. Given any number $\varepsilon > 0$, there exists a number $\delta > 0$ such that, if an initial datum $\Psi^0 \in \mathcal{E}(\mathbb{R}^N)$ satisfies the condition

$$d(\Psi^0, U_p) < \delta,$$

then the corresponding solution $\Psi$ for (NLS) satisfies

$$\sup_{t \in \mathbb{R}} \inf_{U \in \mathcal{M}(p)} d(\Psi(\cdot, t), U) < \varepsilon.$$
Thank you for your attention!