Travelling waves for the nonlinear Schrödinger equation

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Introduction

1. The nonlinear Schrödinger equations

The nonlinear Schrödinger equations under consideration write as

$$i\partial_t\Psi + \Delta\Psi + \Psi f\left(|\Psi|^2\right) = 0, \tag{NLS}$$

for a function $\Psi : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$ and a nonlinearity $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(1) = 0 \quad \text{and} \quad f'(1) < 0.$$

A typical example is the Gross-Pitaevskii equation given by

$$i\partial_t\Psi + \Delta\Psi + \Psi\left(1 - |\Psi|^2\right) = 0, \tag{GP}$$

for $f(\tau) = 1 - \tau$.

The (NLS) equation is Hamiltonian. Its energy is given by

$$E(\Psi) = \frac{1}{2}\int_{\mathbb{R}^N} \Big(|\nabla \Psi|^2 + V(|\Psi|^2)\Big),$$

where the potential V is equal to

$$V(\tau) = \int_\tau^1 f(\sigma)\, d\sigma.$$

For (GP), this expression is exactly the Ginzburg-Landau energy. At least formally, solutions Ψ with finite energy satisfy a non vanishing condition at infinity

$$|\Psi(x)| \to 1, \qquad \text{as } |x| \to \infty.$$

The (NLS) equation is dispersive. Its linearization around the constant solution $\Psi = 1$ leads to the linear equation

$$i\partial_t \varepsilon + \Delta \varepsilon + 2f'(1)\mathrm{Re}(\varepsilon) = 0.$$

The dispersion relation is equal to

$$\omega^2 = -2f'(1)|k|^2 + |k|^4.$$

The sound speed is

$$c_s = \sqrt{-2f'(1)}.$$

The (NLS) equation also owns an hydrodynamical formulation. When the solution Ψ can be expressed in terms of the Madelung transform

$$\Psi = \sqrt{\rho}\, e^{i\theta},$$

the hydrodynamical variables $(\rho, v = 2\nabla\theta)$ satisfy the hydrodynamical system

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho\, v) = 0, \\ \partial_t v + v \cdot \nabla v - 2\nabla f(\rho) = 2\nabla\Big(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\Big). \end{cases} \tag{HGP}$$

2. Travelling waves

Travelling waves are special solutions of the form

$$\Psi(x,t) = U_c(x_1 - ct, \dots, x_N),$$

for speeds $c \in \mathbb{R}$. Their profile U_c is solution to the nonlinear elliptic equation

$$-ic\partial_1 U_c + \Delta U_c + U_c f\Big(|U_c|^2\Big) = 0. \qquad (1)$$

In dimension $N = 1$, the (non constant) travelling waves for (GP) are uniquely given by

$$U_c(x) = \sqrt{\frac{2-c^2}{2}} \tanh\left(\frac{\sqrt{2-c^2}}{2} x\right) + i \frac{c}{\sqrt{2}},$$

for $|c| < \sqrt{2}$ (up to translation and phase shift).

Due to the integrability of the equation by means of the inverse scattering transform (Zakharov-Shabat [73]), the dark solitons U_c are believed to play a major role in the long-time dynamics.

I. The Cauchy problem

Theorem (Zhidkov [01], Gérard [06], Killip-Oh-Pocovnicu-Visan [12]).
For $N \in \{1, 2, 3, 4\}$, *let* Ψ_0 *be a function in the energy set*

$$\mathcal{E}(\mathbb{R}^N) = \Big\{ \Psi : \mathbb{R}^N \to \mathbb{C} \text{ s.t. } E(\Psi) < +\infty \Big\}.$$

There exists a unique global solution in $\mathcal{C}^0(\mathbb{R}, \mathcal{E}(\mathbb{R}^N))$ *with initial datum* Ψ_0 *for* (GP). *Moreover, the Ginzburg-Landau energy is conserved along the flow.*

See Gallo [04, 06] for (NLS).

II. Construction of (non constant) travelling waves

For (GP), Jones, Putterman and Roberts [82, 86] investigated the existence and qualitative properties of travelling waves in dimensions $N = 2$ and $N = 3$.

They claimed the non-existence of supersonic travelling waves and exhibited a smooth branch of subsonic travelling waves.

1. Non-existence of travelling waves

Theorem (Bethuel-Saut [99], G. [03, 04]).
For (GP)*, a travelling wave with speed* $c = 0$ *for* $N \geq 2$, $c > \sqrt{2}$ *for* $N \geq 2$*, and* $c = \sqrt{2}$ *for* $N = 2$*, is constant.*

See Maris [08] for (NLS).

Theorem (Bethuel-G.-Saut [07], de Laire [08]).
Let $N \geq 3$*. For* (GP)*, there exists a number* $\mathcal{E}_N > 0$ *such that a travelling wave* U *with energy*

$$E(U) \leq \mathcal{E}_N,$$

is constant.

2. Minimizing travelling waves

For fixed $\mathfrak{p}$, minimizing travelling waves $U_{\mathfrak{p}}$ solve the variational problem

$$E_{\text{min}}(\mathfrak{p}) = \inf \Big\{ E(u), u : \mathbb{R}^N \to \mathbb{C} \text{ s.t. } p(u) = \mathfrak{p} \Big\}.$$

Here, the scalar momentum p is the first component of the momentum, which is formally given by

$$p(u) = \frac{1}{2} \int_{\mathbb{R}^N} \langle i \partial_1 u, u \rangle_{\mathbb{C}}.$$

The speed $c_{\mathfrak{p}}$ is the Lagrange multiplier of the previous problem.

Theorem (Bethuel-G.-Saut [07]).

(i) *For* $N = 2$ *and any* $\mathfrak{p} > 0$, *there exists a minimizing travelling wave* $U_{\mathfrak{p}}$ *for* (GP).

(ii) *For* $N = 3$, *there exists a critical value* $\mathfrak{p}_* > 0$ *such that there exists a minimizing travelling wave* $U_{\mathfrak{p}}$ *for* (GP) *if and only if*

$$\mathfrak{p} \geq \mathfrak{p}_*.$$

(iii) *The speed* $c_{\mathfrak{p}}$ *of the travelling wave* $U_{\mathfrak{p}}$ *satisfies*

$$0 < \frac{dE_{\min}}{d\mathfrak{p}}\big(\mathfrak{p}^+\big) \leq c_{\mathfrak{p}} \leq \frac{dE_{\min}}{d\mathfrak{p}}\big(\mathfrak{p}^-\big) < \sqrt{2}.$$

See Chiron-Maris [17] for (NLS).

Theorem (Chiron-Pacherie [19, 19, 21]).

(i) *Let* $N = 2$. *There exists a number* $\mathfrak{p}_0 > 0$ *such that, for any* $\mathfrak{p} \geq \mathfrak{p}_0$, *the travelling waves* $U_\mathfrak{p}$ *for* (GP) *are unique (up to translation and phase shift). Moreover, they form a smooth branch of travelling waves.*

(ii) *In the limit* $\mathfrak{p} \to 0$, *their speed* $c_\mathfrak{p}$ *is of order*

$$c_\mathfrak{p} \sim 2\pi/\mathfrak{p},$$

and they own two vortices of degree ± 1 *at a distance* $d_\mathfrak{p}$ *of order*

$$d_\mathfrak{p} \sim \frac{\mathfrak{p}}{\pi}.$$

See also Bethuel-Saut [99], Bethuel-Orlandi-Smets [04] and Chiron [04].

Set $\Omega_L = \mathbb{R} \times \mathbb{T}_L$ for any number $L > 0$.

Theorem (de Laire-G.-Smets [22]).
(i) *Let* $0 < \mathfrak{p} \leq \pi/2$. *There exists a number* $L_{\mathfrak{p}} > 0$ *such that, for any* $0 < L \leq L_{\mathfrak{p}}$, *there exists a minimizing travelling wave* $U_{\mathfrak{p}L}$ *for* (GP) *on* Ω_L.
(ii) *The travelling wave* $U_{\mathfrak{p}L}$ *only depends on the variable* x_1. *In particular, it is a dark soliton.*

3. Subsonic travelling waves

Theorem (Maris [09]).
Let $N \geq 3$. *There exists a non constant travelling wave* U_c *of* (NLS) *for any speed* $0 < c < \sqrt{2}$.

See also Bellazzini-Ruiz [19].

III. Orbital and asymptotic stability of travelling waves

1. In dimension $N = 1$

a. Orbital stability of dark solitons

Let us endow the energy set $\mathcal{E}(\mathbb{R})$ with the distance

$$d_c(\psi_1, \psi_2)^2 = \int_{\mathbb{R}} |\psi_2' - \psi_1'|^2 + (1 - |U_c|^2)|\psi_2 - \psi_1|^2 + \Big||\psi_1|^2 - |\psi_2|^2\Big|^2.$$

Theorem (Bethuel-G.-Saut [08], Bethuel-G.-Saut-Smets [08]).
Let $c \in (-\sqrt{2}, \sqrt{2})$*. There exist two numbers* $\delta_c > 0$ *and* $K_c > 0$ *such that, if an initial datum* $\Psi^0 \in \mathcal{E}(\mathbb{R})$ *satisfies the condition*

$$\delta := d_c(\Psi^0, U_c) < \delta_c,$$

then there exist two functions $a \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ *and* $\theta \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$*, with*

$$\sup_{t \in \mathbb{R}} |a'(t) - c| < K_c \, \delta,$$

such that the corresponding solution Ψ *for* (GP) *satisfies*

$$\sup_{t \in \mathbb{R}} d_c\Big(e^{-i\theta(t)} \Psi(\cdot + a(t), t), U_c\Big) < K_c \, \delta.$$

See also Gérard-Zhifei Zhang [08], and Chiron [13] for (NLS).

b. Asymptotic stability of dark solitons

Theorem (Bethuel-G.-Smets [13], G.-Smets [14]).
Let $c \in (-\sqrt{2}, \sqrt{2})$. *There exists a number* $\delta_c > 0$ *such that, if the initial datum* $\Psi^0 \in \mathcal{E}(\mathbb{R})$ *satisfies the condition*

$$d_c(\Psi^0, U_c) < \delta_c,$$

then there exist a number $c_\infty \in (-\sqrt{2}, \sqrt{2})$, *and two functions* $a \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ *and* $\theta \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$, *with*

$$a'(t) \to c_\infty \quad \text{and} \quad \theta'(t) \to 0,$$

as $t \to +\infty$, *such that the corresponding solution* Ψ *for* (GP) *satisfies*

$$e^{-i\theta(t)}\Psi(\cdot + a(t), t) \to U_{c_\infty} \quad \text{locally uniformly on } \mathbb{R}.$$

See also Cuccagna-Jenkins [16].

2. In higher dimensions

For $N = 2$ and $N = 3$, the energy set $\mathcal{E}(\mathbb{R}^N)$ can be endowed with the distance

$$d(f,g) = \big\| f - g \big\|_{L^2(B(0,1))} + \big\| \nabla f - \nabla g \big\|_{L^2(\mathbb{R}^N)} + \big\| |f|^2 - |g|^2 \big\|_{L^2(\mathbb{R}^N)}.$$

Theorem (Chiron-Maris [11]).
Let $\mathcal{M}_\mathfrak{p}$ be the set of minimizing travelling waves $U_\mathfrak{p}$ with scalar momentum $\mathfrak{p}$ (with $\mathfrak{p} \geq \mathfrak{p}_*$ if $N = 3$).
Fix a travelling wave $U_\mathfrak{p} \in \mathcal{M}_\mathfrak{p}$. Given any number $\varepsilon > 0$, there exists a number $\delta > 0$ such that, if an initial datum $\Psi^0 \in \mathcal{E}(\mathbb{R}^N)$ satisfies the condition

$$d(\Psi^0, U_\mathfrak{p}) < \delta,$$

then the corresponding solution Ψ for (NLS) *satisfies*

$$\sup_{t \in \mathbb{R}} \inf_{U \in \mathcal{M}(\mathfrak{p})} d\Big(\Psi(\cdot, t), U\Big) < \varepsilon.$$

Thank you for your attention!